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Masili, Gustavo
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24. July 2006

Online at <http://mpra.ub.uni-muenchen.de/2134/>
MPRA Paper No. 2134, posted 07. November 2007 / 02:14

Auction with aftermarket for budget constrained bidders

Gustavo Santos Masili

Departament d'Economia i d'Història Econòmica

Universitat Autònoma de Barcelona

08193 Bellaterra (Barcelona), Spain

July 2006

Abstract

The paper compares different auction formats for sale of a single patented innovation for budget constrained bidders. This unit decreases the marginal cost of production in the aftermarket for its owner by an amount which depends on the money invested on the development of this technology. As the bidders have a fixed budget that must be used to pay the final auction price and also to develop the new technology, the winner has incentives to pay a low amount for his unit to increase the amount available to invest in cost reduction. Conversely the loser has incentives to induce induce a higher price to be paid by the winner in order to increase aftermarket profits. This conflict of interest generates the willingness to pay (WTP) for the patent through an endogenous process, which may end up by stablishing a higher WTP for the lowest financed firm. Given this background, the case in which the players have different initial budgets may generate multiple equilibria for all studied auction mechanisms. These equilibria produce different consumer surplus and, thus, a central government with an anti-trust behavior is able to choose the auction that generates the refined equilibrium leading to the highest consumer surplus.

JEL Classification: C72, D44, L11.

Keywords: Market design, auction, aftermarket, budget constraints, investment.

Supervisor: Roberto Burguet

1 Introduction

Auction is a vetust trading device defined by several rules intending to specify an efficient market price for goods with unknown value. For such achievement it is imperative to design the mechanism that propitiates the auctioneer's eagerness according to the market conditions.

This paper considers the auctioning of a patented innovation which brings technological advantage for its owner in the Cournot aftermarket by reducing the marginal production cost. Needless to say that this problem itself would represent a traditional unconstrained optimization for the aftermarket preceded by an independent decision for the auction. However the model considers budget constrained agents who are able to invest on the patent development a value corresponding to what remained after purchasing the patent on the auction. This fact modifies the behavior of the players on the auction process, transforming a usually exogenous mechanism into an endogenous one.

Traditionally agents don't bid a value higher then their willingness to pay for the auctioned good (See Krishna, 2002 and Burguet, 2000). Nevertheless in the studied case this strategy may be used since the loser on the auction has incentives to increase the price paid by the winner in order to decrease the investment applied in cost reduction and, consequently, increase his own profit in the aftermarket.

One of the consequences of this endogenous bidding process is the possibility of multiplicity of Nash Equilibrium (NE) prices for any traditional auction format whenever budget asymmetry is considered¹. By using game theoretical tools it turns out that some equilibria seem more predictable then others. Then by choosing the auction format it is possible to generate different consumer surplus related to the refined equilibrium produced. One of the basic findings in the Cournot model where the two players have different marginal cost is that the consumer surplus and the marginal cost of the low-cost firm are negatively correlated (see Shapiro, 1989). Conversely Schwartz (1989) identified necessary conditions for which lowering equilibrium price also reduced welfare. He argues that in an oligopoly, part of a firm's gain is due to altering the market equilibrium in its favour and to the disadvantage of the rivals. A possible mean for this achievement is investing in cost reduction, which can reduce welfare by diminishing the equilibrium output. In the model presented,

¹Pitchik (2006) concludes that in the presence of budget constraints for sequential auctions there may exist more then one symmetric equilibrium functions differnig with respect to allocation, prices and revenue.

given that the players are budget constrained, the more one pays for the patent represents less budget that may be devoted for investment and, hence, the smaller the consumer surplus. But a new feature considered is that the amount paid for the patent acquisition also counts as part of the consumer surplus provided that a Government Authority is the responsible for its administration. In this case there is the trade-off between increasing the consumer surplus by changing the auction price and the investment in cost reduction. As consequence, an anti-trust regulator may force the auctioneers to use a given auction format that maximizes the consumers satisfaction.

Additionally, this endogeneity may generate inefficient allocation for the patented innovation. This may occur due to the higher incentives for the weaker bidder to avoid that his opponent wins because the losses considering the amount that he pays are less expensive than the ones brought by the aftermarket losses whenever his opponent wins. Also the stronger competitor prefers to avoid winning after a certain patent price given that his losses considering that his opponent wins are smaller than the amount that he must pay for the patent. Che and Gale (1998) study auctions when bidders face an increasing marginal cost expenditure. They evaluate a model with two dimensional private information in which buyers are distinguished by their willingness to pay and ability to pay (financial constraints). Their model occasionally also generates inefficient allocations since the good may not be acquired by the player with highest valuation, but may instead go to a better-financed one with lower valuation. Maskin (1992), and Shleider and Vishny (1992) also recognize the possibility that a good may not be allocated to the highest-valuation buyer.

The current paper demonstrates the existence of conditions for the English auction, which produces the higher equilibrium price, to outperform the Dutch one from the consumer's point of view. The extension of this result for the sealed-bid format is straightforward² given that there are two bidders (which represents the inexistence of signaling for the English auction) and the information is complete (see Krishna, 2002). Consequently the Second Price Auction (SPA) outranks the First Price Auction (FPA). Che and Gale (1998) found the contrary result when considering their model with private information and exogenous valuation.

Auctions of licenses with constrained bidders are also studied by Pagnozzi (2005). In his model weak and strong bidders compete for an auctioned good and the winner has the option of reselling his item to the loser. In this sequential auction environ-

²Goerre (2003) shows that in an incomplete information game with signaling the English auction is not strategically equivalent to the Second Price Auction as the cost of signaling is higher for the English format given that the player incurs the total cost of his signal.

ment it is usually expected that the strong bidder will increase the price in order to weaken its competitor, and this behavior would induce weak competitors not to have incentives to participate on the auction since they would get null profit. But when the budget constraints matter the author mentions that the strong bidder is in a better bargaining position in the resale market if the weak bidder pays a low rather than high price for the good. This induces the weak bidder to bid more aggressively to increase his profits as he knows that he will win the first auction. This happens because the wealth constrained bidder enjoys limited liability and treats the auction prize as an option (whenever the project seems unprofitable, he may declare bankruptcy and lose his wealth). Burguet and McAfee (2005) consider an auction with budget constrained firms where the winners for the n auctioned goods have the right to compete in a Cournot market. The price paid on the auction influences the maximum possible production cost of the firm in the aftermarket. This means that if the auction prices are high enough, firms are not able to invest the optimal amount in the deployment of services. Goerre (2003) also considers the case of an auctioned patent with Cournot aftermarket, but in his case he considers signalling with private information, and a fix marginal cost reduction for the owner of the innovation. Furthermore Benoit and Krishna (1998) demonstrate that budget constrained bidding may be the result of conscious choice rather than of exogenous factors as liquidity constraints or capital market imperfections for multiple-object auctioning.

The correlation between investment and consumer surplus is studied by Bandulet and Morasch (2003), who consider an heterogeneous good duopoly with quantity competition. In their model there is a local firm, with no transport and cost, and a distant one that may invest in transport cost reduction. The paper compares private and social planner's investment, arguing that the investing firm neither considers the positive impact on consumers nor the negative effect on its competitor. They find out that firms overinvest relative to the social optimum, which means that the negative impact on its competitor exceeds the gain in consumer surplus.

The paper is organized as follows. The general framework in which two firms compete for a patent innovation that reduces its owner's cost for the Cournot market according to the amount invested on its development is evaluated in Section 2. Section 3 describes the bidding strategies related to each of the studied auction formats whenever firms have different initial budget. Considering these results Section 4 characterizes the welfare related to the usage of each auction mechanism and, given few conditions, establishes the English auction as the preferred format for an anti-trust authority responsible for its design. Section 5 states the implications of the findings and proposes extensions.

2 The model

Consider a Cournot duopoly with risk neutral firms producing at constant marginal cost c and facing an aggregate demand function, i.e. $P(Q) = a - bQ$, where $Q = q_1 + q_2$ is the total supply of this industry. Now suppose a patented innovation, which may reduce the marginal cost of production for its owner, is sold through an auction at price θ . The buyer chooses an investment I to develop this patent and, depending on the amount invested, the firm has a different marginal cost of production given by the function $c(I)$, which is strictly decreasing and convex³. In order to make optimal decisions firms must consider their initial budget B_i for $i = 1, 2$, that may be used to pay for the patent and also for investment in cost reduction. The timing of this market interaction is displayed in table 1.

Table 1: Timetable (3-stage game):

1st stage (Auction)	2nd stage (Investment)	3rd stage (Cournot market)
<i>Winner</i> <i>(Pays θ for patent)</i>	<i>Winner chooses I</i> <i>($I \leq B - \theta$)</i>	<i>1 chooses $q_1^*(I)$</i>
<i>Loser</i> <i>(Pays 0)</i>		<i>2 chooses $q_2^*(I)$</i>

The following two cases will be analyzed in the paper: (i) both players have the same budget; and (ii) their budget is different. The former enables the understanding of the basic features of the model while the latter is the case in which the auction format will have consequences on the outcome of this market.

2.1 Both players have the same budget constraint

The first model evaluated considers the case in which both players are completely symmetric with $B_1 = B_2 \equiv B$. The present problem is analysed by backward induction.

³Flaherty (1980) mentions that it is commonly assumed that extra cost reduction becomes more expensive to acquire as marginal cost reduction increases.

2.1.1 3rd stage: Cournot market

Assume w.l.o.g. that player 1 buys the patent in the first stage and that θ is the price he pays for it. In the second stage he has invested I on patent development for reduction of the marginal cost of production. This results in a Cournot game where firms 1 and 2 have marginal cost $c(I)$ and c respectively. Thus

$$\Pi_1^*(I) = \max_{q_1} P(Q) q_1 - q_1 c(I) - I - \theta, \quad (1)$$

$$\Pi_2^*(I) = \max_{q_2} P(Q) q_2 - q_2 c, \quad (2)$$

where $P(Q)$ is the continuous and differentiable inverse demand function. To ensure that this problem is strictly concave so that the Lagrange method will lead to a unique optimal interior ($q_i > 0$) solution, it is assumed that $P''(Q) \cdot Q + 2P'(Q) < 0$.

Considering the demand elasticity given by $\varepsilon \equiv -\frac{P}{Q} \frac{\partial Q}{\partial P}$ and denoting the market share of player i as $S_i \equiv \frac{q_i}{Q}$ it is possible to write the FOC as

$$P(Q) \left(1 - \frac{1}{\varepsilon} S_1\right) - c(I) = 0, \quad (3)$$

$$P(Q) \left(1 - \frac{1}{\varepsilon} S_2\right) - c = 0. \quad (4)$$

Equations 3 and 4 determine respectively q_1^* and q_2^* both as a function of I .

2.1.2 2nd stage: Investment (for player 1)

The only player that participates in the second stage is the buyer of the patent in the first stage. The optimal investment for firm 1 is the value that maximizes his profits considering his budget constraint and the price θ paid for the patent on the first stage.

$$I^* \in \arg \max_{I \leq B - \theta} \Pi_1^*(I). \quad (5)$$

It is possible to write the FOC of this problem:

$$\frac{\partial \Pi_1}{\partial I} = \frac{\partial c(I)}{\partial I} \left\{ \frac{\partial q_1}{\partial c(I)} \left[P(Q) \left(1 - \frac{1}{\varepsilon} S_1 \right) - c(I) \right] + q_1 \left(\frac{\partial P(Q)}{\partial Q} \frac{\partial q_2}{\partial c(I)} - 1 \right) \right\} - 1 \geq 0,$$

where the term in brackets is zero given the Cournot FOC. Rewriting the optimality condition

$$\frac{\partial \Pi_1}{\partial I} = q_1 \frac{\partial c(I)}{\partial I} \left(\frac{\partial P(Q)}{\partial Q} \frac{\partial q_2}{\partial c(I)} - 1 \right) - 1 \geq 0.$$

For this stage, two cases may occur when considering the FOC:

i. the budget constraint is not binding ($I < B - \theta$) From the FOC, whenever the Lagrange multiplier $\lambda = 0$ (constraint is not binding):

$$\frac{\partial \Pi_1}{\partial I} = q_1 \frac{\partial c(I)}{\partial I} \left(\frac{\partial P(Q)}{\partial Q} \frac{\partial q_2}{\partial c(I)} - 1 \right) - 1 = 0. \quad (6)$$

Solving equation 6 leads to the optimal investment I_{NB}^* for the case where the budget constraint does not affect the Cournot market outcome. For the moment it is assumed that there is a unique solution for this equation.

ii. the budget constraint is binding ($I = B - \theta$) From the FOC, whenever the Lagrange multiplier $\lambda > 0$ (constraint is binding):

$$\frac{\partial \Pi_1}{\partial I} = q_1 \frac{\partial c(I)}{\partial I} \left(\frac{\partial P(Q)}{\partial Q} \frac{\partial q_2}{\partial c(I)} - 1 \right) - 1 > 0. \quad (7)$$

Equation 7 indicates that the marginal profit of player 1 with respect to his investment is still increasing at the optimal investment. Thus it turns out that the solution is not interior and therefore it must be the case that $I_B^* = B - \theta$. So it is possible to write a unique function that specifies the optimal investment for any B

$$I^* = \min \{ I_{NB}^*, I_B^*(\theta, B) \}.$$

2.1.3 1st stage: Determining the willingness to pay

For the moment the discussion will abstract from the auction mechanism properties and focus on price and willingness to pay.

Definition 1 *The willingness to pay θ_i for player i , $i \in \{1, 2\}$, is the price for the patent on the first stage of the game that makes player i indifferent between buying the patent or having the competing firm buy the patent at that price.*

These values are investigated considering the two possible cases that the participants may be facing: (i) budget constraint is not binding, and (ii) budget constraint is binding. These cases occur when the optimal investment given the initial budget is respectively below or equal to the one considering unconstrained bidders.

i. the budget constraint is not binding For the moment, assume B is infinite so that firms are not bounded when investing. In this case, both his investment in case he wins or the investment of his rival in case he loses are independent of the price paid. The willingness to pay of player i for the case in which the constraint is not binding for him, θ_{NB}^* , is calculated by equating the payoff when he wins, Π_i^W , with the one in case he loses, Π_i^L , for the optimal values,

$$\Pi_i^{W*}(I_{NB}^*, \theta) - \Pi_i^{L*}(I_{NB}^*) = 0 \rightarrow \theta_{NB}^*.$$

ii. the budget constraint is binding Now assume $I = B - \theta$. The willingness to pay for player i for the case in which the constraint is binding for him, θ_B^* , is calculated by equating Π_1 and Π_2 , but this time using:

$$\Pi_i^{W*}(I_i^* = B - \theta) - \Pi_i^{L*}(I_j^* = B - \theta) = 0 \rightarrow \theta_B^*(B).$$

This willingness to pay θ_B^* increases with budget increments until the constraint becomes not binding; at this point, θ_{NB}^* becomes constant with respect to budget changes.

Definition 2 $B^* \equiv \theta_{NB}^* + I_{NB}^*$. *It is the minimum initial budget for a player at which his budget constraint does not bind.*

An important issue to guarantee the stability of equilibrium for any initial budget for the players is that this function $\theta(B)$ is continuous at the value of B^* . Thus the following lemma guarantees this continuity property and its proof is formalized on the appendix.

Lemma 1 *The function $\theta(B)$ that is a mapping $\theta : B \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is strictly increasing for $B < B^*$ and continuous at the value of B^* for which the constraint becomes not binding.*

Proof. See appendix. ■

For symmetric budgets the following proposition is straightforward and no proof is provided.

Proposition 1 *The unique pure-strategy equilibrium outcome of either a FPA, SPA, English or Dutch auction is $\theta = \theta^*$.*

As the willingness to pay θ^* is the unique NE for the case in which both players have the same initial budget, any traditional auction mechanism would lead to the same patent price and, thus, the same consumer surplus. As it will later be shown, the additional feature of having different initial budgets that is presented in the next section gives the problem multiple Nash Equilibria.

2.2 Asymmetric budgets: $B_1 > B_2$

Now it is assumed that players 1 and 2 have different budget constraints B_1 and B_2 respectively. Again, consider the buyer i pays θ for the patent and on the second stage of the model he invests I_i on the patent for reduction of the marginal cost of production. After deciding the optimal investment, firms compete à la Cournot during the third stage. There are three possibilities: (i) Players 1 and 2 have their budget constraints not binding; (ii) Players 1 has the constraint not binding and player 2 has it binding; (iii) Players 1 and 2 have the constraints binding.

In each of them the budget constraint only influences the initial stage of the game. Indeed, a player must evaluate his willingness to pay for the patent considering that in the second stage he will invest all the remaining budget whenever the

optimal investment is inaccessible due to budget constraint or the optimal investment considering the gains of trade generated on the third stage by the reduction on marginal cost. Hence the second and third-stage optimal strategies of the game remain as in the previous section.

2.2.1 Budgets don't bind

This is the case in which both players have the initial budget above B^* . This implies that any of them would be able to invest the optimal unconstrained amount in the cost reduction during the second stage of the game whenever they win the auction. Therefore their payoff functions are identical in case they are the winner (or loser) since the value spent with the patent purchasing process ($I^* + \theta$) is identical. Thus both players still have the same willingness to pay.

2.2.2 Both are budget constrained

This is the case in which both players have their initial budgets below B^* . The willingness to pay for player i is still defined as the price for which $\Pi_i^{W^*} = \Pi_i^{L^*}$. As $B_1 \neq B_2$, it will usually be the case that these values are distinct, because for player 1 both Π^{W^*} and Π^{L^*} are higher than for player 2. The player with the highest willingness to pay is therefore unascertained.

Example: For this case the complete procedure will be exemplified by using the linear inverse demand function $P(Q) = a - bQ$, and the marginal cost reduction function $c(I) = ce^{-I}$. Table 2 represents the solutions for the second and third stages of this game using the backward induction principle for both agents whenever player i is the winner of the auctioned patent.

Table 2: 3^{rd} and 2^{nd} stages of game when $B^* > B_1 > B_2$.

Player i is the winner		
3rd stage	$q_i^* = \frac{a+c-2c(I_1)}{3b}$	$q_j^* = \frac{a-2c+c(I_1)}{3b}$
2nd stage	$I_i^* = B_i - \theta_i$	
Winner	$\Pi_i^{WIN} = b \left(\frac{a+c-2ce^{-B_i+\theta_i}}{3b} \right)^2 - B_i$	
Loser	$\Pi_j^{LOSE} = b \left(\frac{a-2c+ce^{-B_i+\theta_i}}{3b} \right)^2$	

1st stage: Determining the willingness to pay In order to choose the bids for the auction the indifference points between winning and losing the auction for both players must be evaluated. Considering player i :

$$\Pi_i^{WIN} = b \left(\frac{a+c-2ce^{-B_i+\theta}}{3b} \right)^2 - B_i = b \left(\frac{a-2c+ce^{-B_j+\theta}}{3b} \right)^2 = \Pi_i^{LOSE}.$$

From this equality one may obtain the indifference value for player i :

$$\theta_i = \ln \left(-\frac{\sqrt{(2ce^{-B_j}-2ce^{-B_i}-ae^{-B_j}-2ae^{-B_i})^2-3(4e^{-2B_i}-e^{-2B_j})(2ac-3bB_i-c^2)}}{4ce^{-2B_i}-ce^{-2B_j}} + \frac{a(e^{-B_j}+2e^{-B_i})+2c(e^{-B_i}-e^{-B_j})}{4ce^{-2B_i}-ce^{-2B_j}} \right) \quad (8)$$

Equation 8 determines the willingness to pay for both players considering the case in which they are budget constrained⁴. The profit functions for players 1 and 2 are characterized on Figure 1⁵, as well as both indifference values.

In this case, the willingness to pay are usually distinct from each other and, hence, there may exist cases in which player 2 is willing to pay more for the good than player 1 even having less budget. It will later be shown that whenever $\theta_1 \neq \theta_2$ there exist multiple equilibria, which makes the the choice of the auction format to be used an interesting issue.

⁴Note that the willingness to pay may be higher than the total budget that a firm disposes. This may change the outcome of the auctioning process, but nevertheless it is a tool that indicates the intentions of the firm.

⁵All graphical illustrations in the paper consider the linear inverse demand function $P(Q) = a - bQ$, and the marginal cost reduction function $c(I) = ce^{-I}$.

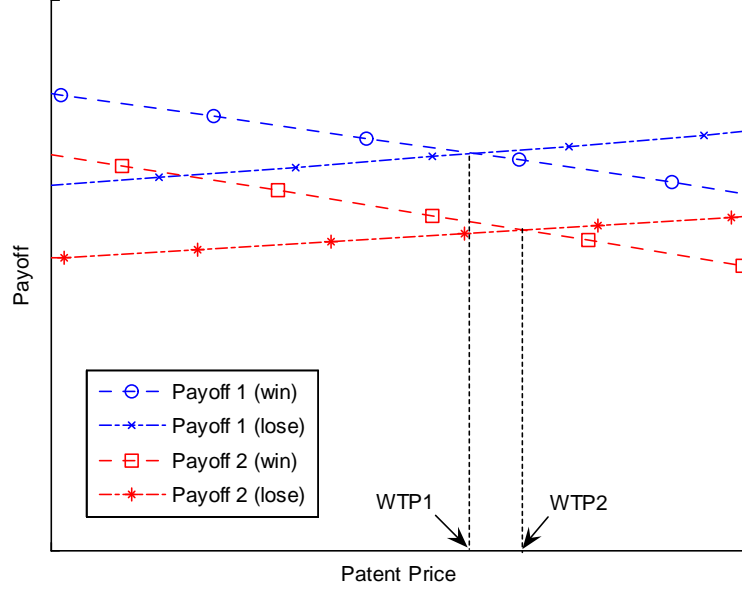


Figure 1: Example of profits whenever both are budget constrained

2.2.3 Player 1's constraint does not bind but player 2's does

In this case player 1 has the initial budget above B^* and player 2 has his initial budget below this value. Note that if player 1 is the winner the profit functions for both players behave as in the previous case, but when player 2 is the winner both profits depend on his initial budget and also on the price paid for the patent.

Hence θ_1 and θ_2 will both depend on B_2 , and the profit functions for both players whenever they are winners or losers in the auction for the patent are illustrated in Figure 2. The willingness to pay for them are the values of θ for which the profit functions for the same player cross each other.

The figure above shows an important feature of this case that is also present in the first case: player 2 is indifferent between any price θ paid by player 1 for the patent whenever he is the loser. This happens because player 1 will always invest his optimal value independently on his initial budget (which is higher then B^*) and this will affect player 2 on the Cournot market on the same way.

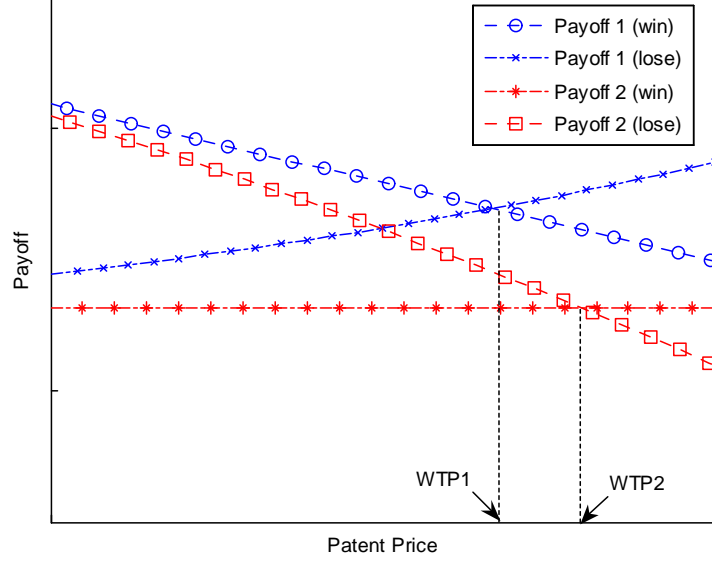


Figure 2: Example of profits whenever only one firm is budget constrained

2.2.4 Bidding strategy for FPA, SPA, English and Dutch auctions

This section mainly describes the expected differences on the final price for the auction considering the traditional oral auctions: English and Dutch. As only two completely informed players participate in the bidding process, these results may be extended to equivalent closed auction formats: Second price auction (SPA) and first price auction (FPA) respectively (see Krishna, 2002 and Burguet, 2000).

To allow for the existence of equilibria and avoid open-set problems for all the auction types it is necessary to implement a tie breaking rule.

Assumption 1: Consider θ_1 and θ_2 as the prices for the patent that makes players 1 and 2 respectively indifferent between winning and losing the auction given their budgets B_1 and B_2 . The tie breaking rule gives the patent to player i whose $\theta_i = \max\{\theta_1, \theta_2\}$ in case of equal bids.

The sequel evaluation consider without loss of generality that $\theta_i = \max\{\theta_i, \theta_j\}$, $i \neq j$, $i, j \in I$, which implies that the tie breaking rule defines player i as the winner in case of bidding draw.

The following proposition uses assumption 1 to obtain the set of NE in pure strategies for all auction formats.

Proposition 2 *All the values $\theta \in [\theta_j, \theta_i]$ are NE strategies for the FPA, SPA, English Auction and Dutch Auction considering the tie breaking rule.*

Proof. See appendix. ■

Proposition 2 implies that for the cases in which both players are budget constrained, or one is so while his opponent is not, there exists multiple equilibria. For the remaining case (both unconstrained) this equilibrium is unique because $\theta_1 = \theta_2$.

Which NE is the most powerful way to predict behavior? For sealed bid auctions, this seems difficult to answer. However, for open formats the sequential nature of bidding can be used to obtain a more precise answer. The following formulations of the open auction will be considered.

Assumption 2: *The English auction starts with the auctioneer asking for the lowest acceptable price $P_1 = 0$, and proceeds by increasing the bid in predetermined increments at each stage until one of the players is not willing to pay the current price. At this moment the auction ends and the good is sold to the remaining player, who pays the price at which his opponent dropped from the auction. Consider b_n as the unique bidder who plays at stage n and that $b_n = i$ when n is odd and $b_n = j$ when n is even, and that the last possible price P_N is sufficiently large that no player would ever think about accepting it. The choice at period n may be to accept or reject the current price, where accept takes the game to the next stage in which the opponent has the same choice set the current price is incremented by ε . If b_n chooses to reject, b_{n-1} wins the game and pays P_{n-1} . Conversely the Dutch auction starts with the auctioneer asking a high price P_1 which is lowered successively until one participant is willing to pay the current price (or the predetermined minimum price $P_N = 0$ is reached). That participant pays the last announced price and receives the good. Consider P_1 sufficiently large that no player would ever think about accepting it and the last possible price being zero. The choice at period n may be to accept or reject the current price, where reject takes the game to the next stage in which the opponent has the same choice set and the current price is decremented by ε . If b_n chooses to accept he wins the game and pays P_n .*

Given this background it is possible to obtain the subgame perfect equilibrium

(SPE) for the English and Dutch auctions whenever both players are budget constrained and all $\theta \in [\theta_j, \theta_i]$ represent NE⁶.

Proposition 3 *Assume that $\theta_i = \max\{\theta_i, \theta_j\}$, $i \neq j$, $i, j \in I$. Given assumptions 1 and 2 the SPE for the English auction is θ_i .*

Proof. At P_N no player accepts the offer due to its high value. Then at P_{N-1} , the other firm must choose between accepting or rejecting the current price given that his opponent will reject at next stage. As $P_{N-1} > \theta_{b_{N-1}}^*$, b_{N-1} prefers to reject the current price. The same choice is taken by firm b_{N-2} at the next stage given that $P_{N-2} > \theta_{b_{N-2}}^*$. Now consider that P_x is the first price that is larger than θ_i . Suppose x is odd ($b_x = i$): then by induction, player i prefers to reject at this stage (and at any higher price). Knowing that i drops at stage x , player j also rejects at stage $x - 1$ because he doesn't want to win the auction and pay a value higher than θ_j . So at stage $x - 2$ player i has to decide whether to accept or reject a price $P_{x-2} < \theta_i$. As θ_i is the willingness to pay for player i he considers profitable to pay any value below it, and so he accepts the current price. Then at stage $x - 2$, knowing that player i will accept on stage $x - 1$, player j also accepts to maximize his final profits. Accordingly both of them accept all prices until it reaches P_1 . Now if x is even ($b_x = j$): at stage $x + 1$ player i rejects given that $P_{x+1} > \theta_i$ and, knowing this, j also rejects at stage x . However i accepts the price at stage $x - 1$ since $P_{x-1} < \theta_i$. Again both of them accept all prices until it reaches P_1 . Therefore the equilibrium price is somehow in $[\theta_i - 2\varepsilon, \theta_i]$ when considering the even and odd values for x . Making ε sufficiently small in order to better represent a continuous case, one can infer that the subgame perfect equilibrium for the English auction is θ_i . ■

Proposition 4 *Assume that $\theta_i = \max\{\theta_i, \theta_j\}$, $i \neq j$, $i, j \in I$. Given assumptions 1 and 2 the SPE for the Dutch auction is θ_j .*

Proof. At $P_N = 0$ any player accepts the offer due to its low value. Then at P_{N-1} , the other firm must choose between accepting or rejecting the current price given that his opponent will accept at next stage. As $P_{N-1} < \theta_{b_{N-1}}^*$, b_{N-1} prefers to accept the current price. The same choice is taken by firm b_{N-2} at the next stage given that $P_{N-2} < \theta_{b_{N-2}}^*$. Now consider that P_x is the first price that is smaller than

⁶Pitchik and Schotter (1988) designed an experiment to confirm that perfect equilibrium is a good predictor of prices for sequential auctions with budget constrained bidders.

θ_j . Suppose x is odd ($b_x = i$): then by induction, at stage $x+1$ (and subsequent lower prices) player j accepts the price as $P_{x+1} < \theta_j$. Then at stage x (and subsequent lower prices) player i prefers to accept because $P_x < \theta_i$. Knowing that i accepts at stage x , player j also rejects at stage $x-1$ because he doesn't want to win the auction and pay a value higher than θ_j . So at stage $n-2$ player i has to decide whether to accept or reject a price $P_{x-2} < \theta_i$. Since he knows that his opponent will reject on next stage he prefers to reject in order to win at stage x and pay a lower price. Then at stage $x-2$ and all previous stages both of them reject all prices until it reaches P_1 . Now if x is even ($b_x = j$): at stage x player j accepts as $P_x < \theta_j$. Knowing this i accepts the price at stage $x-1$ since $P_{x-1} < \theta_i$. However player j rejects at stage $x-2$ given that $P_{x-2} > \theta_j$. Again both of them reject all higher prices until it reaches P_1 . Therefore the equilibrium price is somehow in $[\theta_j, \theta_j + 2\varepsilon]$ when considering the even and odd values for x . Making ε sufficiently small in order to better represent a continuous case, one can infer that the subgame perfect equilibrium for the Dutch auction is θ_j . ■

Considering the relationship described previously it is possible to affirm that whenever both players are budget constrained and $B_1 \neq B_2$, Dutch auction generates a lower expected final price when compared to the English auction. This conclusion contradicts the one described by Krishna (2002) for the case of budget constrained bidders. The reason for such a difference is mainly that in the present case players are asymmetric ($B_1 \neq B_2$) and the loser has incentives to raise the price paid by his opponent whereas in the traditional model this incentive is inexistent.

The next section describes how this different outcomes influence the consumer surplus (CS) and also imposes conditions for auction formats to generate higher CS than others.

2.3 Welfare analysis

Cournot equilibrium maximizes a mixture of industry profits and the social welfare as mentioned by Bergstrom and Varian (1985). This explains the willingness of economists to understand the consequences of changes in both factors. The level of social welfare may change within different market conditions such as the existence of the patent. This section evaluates how different prices of the patent affect the consumers and firms.

The aggregate consumer surplus is the sum of the welfare gained by consumers when they engage trade. Traditionally it involves only the first term on the right hand side of equation 9. For the case of the auction it is assumed that the auctioneer is a government who represents the consumers and therefore all money paid for the patent is reverted to consumers⁷. The difference in consumer's surplus generated by the introduction of the patent on the market is:

$$CS(\theta) = \int_0^{Q^*(\theta)} (P(x) - P(Q^*(\theta))) dx + \theta \quad (9)$$

As it has been shown in last section, it is possible for the auctioneer to choose a mechanism format that generates equilibrium prices at both extremes, θ_i or θ_j , of the range of willingness to pay.

Therefore the study of the consumer surplus generated by final prices being one of these extreme values may provide information regarding the socially preferred auction format. Of the three cases studied, the paper concentrates on the one when both players are budget constrained⁸.

This case generates two possibilities:

- $\theta_i = \theta_j$: for this case there is only one equilibrium point considering all auction mechanisms and is therefore less interesting for analysis;
- $\theta_i > \theta_j$: this is the case in which player i is willing to pay more for the patent than player j ;

In the last case, an English auction is the socially best alternative when the CS is strictly monotone in θ . Now the slope of $CS(\theta)$ is investigated

⁷Another possible explanation for the addition of the price paid for the patented innovation on the consumer surplus is that the more the agents pay for this innovation, the more money the companies have to invest in research.

⁸If both are not budget constrained there is a unique NE and thus any auction mechanism leads to the same final price. Whenever one is budget constrained while his opponent is not, it is possible to prove that even existing multiple equilibria, the refined equilibrium will be the same if $\theta_1 > \theta_2$ for $B_1 > B_2$. This happens because the loser in the unconstrained situation receives the same final payoff independently on the price paid for the patent by his opponent. However this case is interesting whenever $\theta_1 < \theta_2$ for $B_1 > B_2$ and follows the same reasoning as the studied case.

Lemma 2 *In order to have monotonicity of $CS(\theta)$ whenever both players are budget constrained it must be the case that $(2a - c)^2 - 36b < 0$. Moreover there exists $\hat{\theta} \equiv B - \ln \frac{2c}{2a-c}$ such that for $\theta < \hat{\theta}$ ($\theta > \hat{\theta}$) CS is concave (convex).*

Proof. See appendix. ■

For the case of monotone $CS(\theta)$, low elasticity of demand and/or low intersect a propitiates conditions for an increasing function $CS(\theta)$, which makes the English auction a preferred auctioning mechanism as proven in proposition 4.

Proposition 5 *If $CS(\theta)$ is monotonous the English auction maximizes the consumer surplus and therefore is the preferred format for the central government.*

Proof. See appendix. ■

Whenever monotonicity is not guaranteed there may be circumstances in which the Dutch auction generates a higher CS. Moreover it is possible that the patent price that generates the highest expected CS is between θ_i and θ_j , which means that a Dutch auction with reserve price (equal to this optimal price) should be the elected auction format.

2.4 Conclusions

Auctions are economic mechanisms used to sell goods in a wide variety of markets. Traditional studies of auctions consider that the players are able to pay their optimal bid and also that the auction is an independent event. The example mentioned in this paper refers to budget constrained firms competing on an auction for a patented innovation that may give competitive advantages to its owner depending on the amount invested on its development. This generates the problem of endogenous determination of the willingness to pay, which depends on the initial budget of the participants and also on subsequent aftermarket behavior.

For some of the studied cases, including the one with symmetrically budgeted firms, the auction design does not influence the final equilibrium because there is a unique pure strategy that satisfies Nash Equilibrium conditions among standard formats. On the other hand there are few cases (i.e. when both firms are budget

constrained) in which there are multiple equilibria in pure strategies that may occur. Some of these equilibria are more predictable than others depending on the auction format elected by the auctioneer. The ascending open auction picks the highest non-rationing price while the Dutch one defines the lower equilibrium value as the patent's price. This outcome is related to the fact that firms may have incentives to bid above their willingness to pay given this market setting.

For cases in which a central authority is responsible for the auction format choice, given his anti-trust position, it seems reasonable to elect the one that leads to a higher aggregate consumer surplus. Considering the traditional concept of the consumer surplus⁹ one can infer that increasing the price paid during the auction is negatively related to the CS and, therefore, the Dutch auction format should be preferred. But when this price enters the CS formulation, it is possible to characterize a monotonicity condition that guarantees the English auction as the optimal social choice. Therefore it is crucial to understand the interdependence of the model in order to define an auction format that propitiates outputs that agree with the auctioneer's intentions.

3 Acknowledgment

I gratefully acknowledge the precious help and support from my supervisor, Roberto Burguet. Finally, I am grateful to IGSOC for the financial support. All errors are my own responsibility.

4 Appendix

4.1 Lemma 1

Proof.

The function of the final price on the auction given the budget constraint of the players is given by:

$$\theta(B) = \begin{cases} \theta_B^*(B), & \text{if } B < B^* \\ \theta_{NB}^*, & \text{if } B \geq B^*. \end{cases}$$

⁹ $CS(\theta) = \int_0^{Q^*(\theta)} (P(x) - P(Q^*(\theta))) dx.$

By definition of continuity, the function $\theta(B) : [0, \infty) \subset \mathbb{R} \rightarrow \mathbb{R}$ is continuous at B^* if

$$\lim_{B \rightarrow B^{MIN}} \theta_B^*(B) = \theta_{NB}^*.$$

The first step of the proof consists on determining the monotonicity of $\theta_B^*(B)$ for $B \in (0, B^*)$, being the implicit function theorem a tool for this evaluation. It states that given $F : M \times N \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ with $M \subset \mathbb{R}$ and $N \subset \mathbb{R}$ open sets. Suppose (i) $F \in C^p$, $p \geq 1$ and $(B^*, \theta_{NB}^*) \in M \times N$ such that (ii) $F(B^*, \theta_{NB}^*) = 0$ and (iii) $\det \frac{\partial F(B^*, \theta_{NB}^*)}{\partial \theta} \neq 0$. Then $\exists U \subset M$ open with B^* in U , $\exists V \subset N$ open with θ_{NB}^* in V and $\exists ! \theta : U \rightarrow V$ such that $F(B, \theta(B)) = 0$, $\forall B \in U$, $\theta \in C^p(U)$ and

$$d\theta(B) = - \left(\frac{\partial F(B^*, \theta_{NB}^*)}{\partial \theta} \right)^{-1} \frac{\partial F(B^*, \theta_{NB}^*)}{\partial B}, \forall B \in U.$$

For the theorem to be valid:

(i) Construct $F : M \times N \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

$$F(B, \theta) = \Pi^W(\theta - B) - \Pi^L(\theta - B)$$

This function is of class $C^1(D)$ because Π^W and Π^L are continuous functions in θ and B .

(ii) $F(B^*, \theta_{NB}^*) = 0$ by definition of the willingness to pay.

(iii) It turns out that $\frac{\partial F(B^*, \theta_{NB}^*)}{\partial B} = \frac{\Pi^W(\theta - B)}{\partial B} - \frac{\Pi^L(\theta - B)}{\partial B}$. The first term of this equation is positive because if the winning firm disposes a higher budget, there is more money to invest in cost reduction. As this is the case of budget constrained firms, his profits are strictly increasing with investment. The opposite happens to the last term of the equation: profits of the losing firm decrease as the budget increases because, as this implies more investment of the winning firm, the costs of the winning firm become lower on the Cournot market, leading to losses to the losing firm. Hence $\frac{\partial F(B^*, \theta_{NB}^*)}{\partial B} > 0$.

Now $\frac{\partial F(B^*, \theta_{NB}^*)}{\partial B}$ has the contrary influence on the payoffs given that, when the budget is constrained, the more a firm pays for the patent represents less payoff for himself and more for his opponent, leading to $\frac{\partial F(B^*, \theta_{NB}^*)}{\partial B} < 0$. So the condition for inversion of the function $\frac{\partial F(B^*, \theta_{NB}^*)}{\partial \theta}$ is satisfied.

Then $d\theta(B) = - \left(\frac{\partial F(B^*, \theta_{NB}^*)}{\partial \theta} \right)^{-1} \frac{\partial F(B^*, \theta_{NB}^*)}{\partial B} > 0, \forall B \in (0, B^*)$.

It is known that θ_{NB}^* is constant for $B \geq B^*$ and that the profit when $B = 0$ leads to the traditional Cournot profits, which are lower than the one when the winning firm invests the optimal amount. So if continuity is shown, the monotonicity property is also demonstrated. Thus the proof for continuity on B^* is made by contradiction. Suppose $\theta(B)$ is not continuous on B^* ; then it must be the case of (i) $\lim_{B \rightarrow B^*} \theta_B^*(B) > \theta_{NB}^*$ or (ii) $\lim_{B \rightarrow B^*} \theta_B^*(B) < \theta_{NB}^*$. Considering equations 1, 2 and 5:

(i) This implies that for firm 1, $\lim_{B \rightarrow B^*} I = B - \theta_B^*(B) < I_{NB}^*$. But at the limit the investment should be the same given that $\Pi(I)$ is strictly convex and continuous¹⁰ and has the optimal solution at $I = I_{NB}^*$. So $\lim_{B \rightarrow B^*} \theta_B^*(B) \not> \theta_{NB}^*$.

(ii) This implies that for firm 1, $\lim_{B \rightarrow B^*} I = B - \theta_B^*(B) > I_{NB}^*$. But this contradicts the fact that I_{NB}^* is the solution of the unconstrained maximization. Then $\lim_{B \rightarrow B^*} \theta_B^*(B) \not< \theta_{NB}^*$. ■

4.2 Proposition 1

Proof. First price auction

Best response:

For player i :

- For $s_j < \theta_i$: bid $s_i = s_j$;
- For $s_j = \theta_i$: bid $s_i \in [0, s_j]$;
- For $s_j > \theta_i$: bid $s_i \in [0, s_j)$

For player j :

- For $s_i < \theta_j$: not defined;
- For $s_i = \theta_j$: bid $s_j \in [0, s_i]$;
- For $s_i > \theta_j$: bid $s_j \in [0, s_i]$.

Therefore the Nash Equilibria of this game are defined as all strategies $s_i, s_j \in [\theta_j, \theta_i]$ such that $s_i = s_j$.

¹⁰this is the reason why there is a unique solution for the maximization problem for the second stage of the game

Second price auction

Best response:

For player i :

- For $s_j < \theta_i$: bid $s_i \in [s_j, \infty)$;
- For $s_j = \theta_i$: bid $s_i \in [s_j, \infty)$;
- For $s_j > \theta_i$: not defined;

For player j :

- For $s_i < \theta_j$: bid $s_j \in (s_i, \infty)$;
- For $s_i = \theta_j$: bid $s_j \in [s_i, \infty)$;
- For $s_i > \theta_j$: bid $s_j = s_i$.

Therefore the Nash Equilibria of this game are defined as all strategies $s_i, s_j \in [\theta_j, \theta_i]$ such that $s_i = s_j$.

English auction

Game rule: Players bid real numbers subsequently and they are allowed to bid the same value for the tie breaking rule to decide the winner.

Best response:

For player i :

- For $s_j < \theta_i$: bid $s_i = s_j$;
- For $s_j = \theta_i$: bid $s_i = s_j$ or drop auction;
- For $s_j > \theta_i$: drop auction.

For player j :

- For $s_i < \theta_j$: not defined;
- For $s_i = \theta_j$: bid $s_j = s_i$;
- For $s_i > \theta_j$: bid $s_j = s_i$.

Therefore the Nash Equilibria of this game are defined as all strategies $s_i, s_j \in [\theta_j, \theta_i]$ such that $s_i = s_j$.

Dutch auction

Game rule: Players may accept to pay the current real number by bidding it or

they reject this price for the auction process to continue.

Best response:

For player i :

- For $s_j < \theta_i$: bid $s_i = s_j$;
- For $s_j = \theta_i$: bid $s_i = s_j$ or reject;
- For $s_j > \theta_i$: reject.

For player j :

- For $s_i < \theta_j$: not defined;
- For $s_i = \theta_j$: bid $s_j = s_i$;
- For $s_i > \theta_j$: bid $s_j = s_i$.

Therefore the Nash Equilibria of this game are defined as all strategies $s_i, s_j \in [\theta_j, \theta_i]$ such that $s_i = s_j$. ■

4.3 Marginal Consumer Surplus

The following analysis is related to the case in which players have different initial budget and the constraint is binding for both of them. It is considered that the consumer surplus has the price paid for the patent by the winner incorporated to it.

$$CS = \int_0^{Q^*} (P(x) - P(Q^*)) dx + \theta,$$

where $P(Q) = a - bQ$, $Q^* = \frac{2a - c - ce^{-B + \theta^*}}{3b}$. There have been studied the two possible cases:

- Player i is the winner ($\theta_i > \theta_j$) and the equilibrium auction price is θ_i ;
- Player i is the winner ($\theta_i > \theta_j$) and the equilibrium auction price is θ_j .

Matlab simulations for both cases have been taken and the following conclusions were made: There is a positive correlation between the marginal consumer surplus

and the maximum willingness to pay a , given that $a > c$. This may be explained by the fact that increasing a implies on shifting the demand curve up, increasing the total demand Q^* and thus the CS. However changes in the slope of the aggregate demand function b influence the consumer surplus in a negative manner. Therefore the more elastic this aggregate demand function gets, the higher is the consumer surplus. This occurs because as the parameter a is fixed and the slope of the aggregate demand function b decreases, the quantity produced in market equilibrium increases and generates a comparatively smaller decrease in market prices.

4.4 Lemma 2

Proof. Developing the CS:

$$CS = \frac{b}{2} \left(\frac{2a - c - ce^{-I}}{3b} \right)^2 + \theta,$$

where $I = B - \theta$ for the case of binding constraint. Then

$$CS = \frac{b}{2} \left(\frac{2a - c - ce^{-B+\theta}}{3b} \right)^2 + \theta.$$

The FOC requires that $\frac{\partial CS}{\partial \theta} = 0$ for an interior solution

$$\frac{\partial CS}{\partial \theta} = \frac{-ce^{-B+\theta}}{9b} (2a - c - ce^{-B+\theta}) + 1 = 0.$$

After some manipulation

$$\theta = B + \ln \frac{2a - c \pm \sqrt{(2a - c)^2 - 36b}}{2c}.$$

This equation has no real root for $(2a - c)^2 - 36b < 0$.

Now checking the SOC:

$$\frac{\partial^2 CS}{\partial \theta^2} = -\frac{ce^{-B+\theta}}{9b} (2a - c - 2ce^{-B+\theta}).$$

Thus $\frac{\partial^2 CS}{\partial \theta^2} > 0$ (convex CS) implies that $\theta < B + \ln \frac{2a-c}{2c}$ and $\frac{\partial^2 CS}{\partial \theta^2} < 0$ (concave CS) implies $\theta > B + \ln \frac{2a-c}{2c}$. ■

4.5 Proposition 4

Proof. Consider $\theta_i = \max \{\theta_j, \theta_i\}$. For θ_i to be the equilibrium, meaning that the English (also SPA) is the preferred for the central government, it is necessary that $\frac{\partial CS}{\partial \theta} > 0$. So

$$\frac{\partial CS}{\partial \theta} = \frac{-ce^{-B+\theta}}{9b} (2a - c - ce^{-B+\theta}) + 1 > 0.$$

Given the monotonicity property for any $\theta > 0$, it is possible to use $\theta = B$ in order to find a simple condition for such achievement. Then it must be the case in which $9b - 2c(a - c) > 0$. The monotonicity property requires $(2a - c)^2 - 36b < 0 \implies 9b > \frac{(2a-c)^2}{4}$. Thus the condition for the preference over the English auction may also be written as $\frac{(2a-c)^2 - 8c(a-c)}{4} = \frac{(2a-3c)^2}{4} > 0$, which is always true. Therefore given the monotonicity condition it is possible to affirm that the English auction is preferred. ■

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